Real structure in the hyperfinite factor

by

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Introduction. Among the finite factors the hyperfinite one is of main importance. A. Connes has in a series of papers [1,2,3] done penetrating work on this von Neumann algebra, so it is by now very well understood, even though smaller questions remain to be answered. One of them, which he asked me to look into, is how many conjugacy classes of involutory *-anti-automorphisms there are on the hyperfinite factor. Since $B(H)$ - the bounded operators on a complex Hilbert space H - has two conjugacy classes of involutory *-anti-automorphisms, as follows from the classification of irreducible weakly closed Jordan algebras of self-adjoint operators on H, see [7,8], the same might be expected for the hyperfinite factor. However, it will be shown in the present paper that the two classes collapse into one in the hyperfinite factor, so there is only one conjugacy class.

The proof of the above result consists of showing that there is up to conjugacy a unique real von Neumann algebra which generates the hyperfinite factor. Let M denote the hyperfinite factor of type II_1, and let α be an involutory *-anti-automorphism of M. Let $\mathcal{R} = \{x \in M : \alpha(x) = x^*\}$. Then $\mathcal{R}$ is a real von Neumann algebra, i.e. $\mathcal{R}$ is a *-algebra over the reals which is weakly closed and satisfies $\mathcal{R} \cap i\mathcal{R} = \{0\}$ and $\mathcal{R} + i\mathcal{R} = M$. The proof consists of showing that there is an increasing sequence of real subalgebras $\mathcal{R}_n$ of $\mathcal{R}$, with $\mathcal{R}_n$ isomorphic to the real $2^n \times 2^n$ matrices, whose union is weakly dense in $\mathcal{R}$. Then α is the limit of the transpose maps on the $\mathcal{R}_n$'s. In order to find this sequence $\mathcal{R}_n$ we will have to modify the proof of the fundamental theorem of Connes [3, Theorem 5.1], in which he gave several equivalent conditions for a II_1-factor to be hyperfinite. In our proof the factor N in his theorem will be replaced by $\mathcal{R}$, and the relevant operators will be in $\mathcal{R}$ rather than N. It will also be necessary to modify the results of McDuff [5] for our purposes and the classical result of Murray and von Neumann showing the uniqueness of the hyperfinite factor [4, Ch. III, § 7, Théorème 3]. Since a complete proof of all this will be too long we shall first prove the necessary lemmas needed, and from then on just indicate the modifications required in order to prove that the algebra $\mathcal{R}$ defined by α is hyperfinite.

In a recent letter T. Giordano and V. Jones have informed me that they have also shown the uniqueness of the conjugacy classes of involutory *-anti-automorphisms of M, their proof being quite different from mine.

1. Real von Neumann algebras.

Definition. Let $\mathcal{R}$ be a self-adjoint algebra over the reals consisting of bounded operators on a complex Hilbert space H. $\mathcal{R}$ is said to be a real von Neumann algebra if $\mathcal{R}$ is weakly closed and $\mathcal{R} \cap i\mathcal{R} = \{0\}$, where $i\mathcal{R} = \{ix : x \in \mathcal{R}\}$.

By [8, Theorem 2.4] if $\mathcal{R}$ is a real von Neumann algebra then the linear space $\mathcal{R} + i\mathcal{R}$ is a von Neumann algebra. Throughout this section we denote this von Neumann algebra by N.

Lemma 1.1. Let $x \in \mathcal{R}$ and $x = v|x|$ be the polar decomposition of x, with v a partial isometry on H with initial space $\operatorname{supp} x$ and final space $\operatorname{range} x$. Then $v \in \mathcal{R}$.

Proof. Since $v \in N$ there are $u, w \in \mathcal{R}$ with $v = u + iw$, hence $x = u|x| + iw|x| \in \mathcal{R}$. Since $\mathcal{R} \cap i\mathcal{R} = \{0\}$, $w|x| = 0$, and $x = u|x|$. Since $\|x\xi\| = \||x|\xi\|$ for $\xi \in H$, u is an isometry when restricted to $e = \operatorname{supp} x$, hence $u^*u \geq e$. However $e = \operatorname{supp} x = v^*v \in \mathcal{R}$, so $e = v^*v = u^*u + w^*w + i(u^*w - w^*v) = u^*u + w^*w \geq u^*u \geq e$. Thus $w^*w = 0$, and $u = v$. Q.E.D.

Lemma 1.2. Suppose N has no type I portion. Then there exists a real von Neumann subalgebra of $\mathcal{R}$ containing 1 which is isomorphic to the real 2×2 matrices.

Proof. As in the proof of [9, Lemma 2.12] there exist two orthogonal projections e and f in $\mathcal{R}$ with sum 1 and a symmetry $s \in \mathcal{R}$ such that $e = sfs$. Let $v = (e-f)se$. Then $v \in \mathcal{R}$ and satisfies $vv^* = (e-f)ses(e-f) = f$, $v^*v = es(e-f)^2se = e$. Thus e, f, v, v^* form a complete set of matrix units for a I_2-subfactor of N such that the real subalgebra they generate is contained in $\mathcal{R}$ and isomorphic to the real 2×2 matrices. Q.E.D.

Lemma 1.3. Suppose N has a faithful normal semi-finite trace τ, and let e and f be projections in $\mathcal{R}$. Then we have:

(1) if $\tau(eh) = \tau(fh) < \infty$ for all central projections $h \in N$ then there exists a symmetry $s \in \mathcal{R}$ such that $ses = f$.

(2) if there is a partial isometry $v \in N$ such that $v^*v = e$, $vv^* = f$, and there is a projection $g \in \mathcal{R}$ with $g \geq e \vee f$ and $\tau(g) < \infty$, then there is a symmetry $t \in \mathcal{R}$ such that $t(g-e)t = g-f$.

Proof. From the comparison theorem for JW-algebras [11, Theorem 10] there exist a central projection $h \in \mathcal{R}$ and a symmetry $s \in \mathcal{R}$ such that $s(eh)s \leq fh$, $s(f(1-h))s \leq e(1-h)$. Since $\tau(fh) = \tau(eh) = \tau(s(eh)s) \leq \tau(fh) < \infty$, $sehs = fh$ since τ is faithful. Similarly $s(f(1-h))s = e(1-h)$. Adding we obtain $ses = f$, and (1) follows.

If v is a partial isometry in N such that $v^*v = e$ and $vv^* = f$ then $\tau(eh) = \tau(fh)$ for all central projections h in N. The same identity holds for $g-e$ and $g-f$, so by (1) there is a symmetry $t \in \mathcal{R}$ such that $t(g-e)t = g-f$. Q.E.D.

With $\mathcal{R}$ as above the map $\alpha : x+iy \to x^*+iy^*$, $x,y \in \mathcal{R}$, is an involutory $*$-anti-automorphism of N. If N is finite with a separating and cyclic vector then by [9, Theorem 3.8] there exists a conjugation J of H such that $\alpha(x) = Jx^*J$ for $x \in N$. Furthermore if ξ_0 is a separating and cyclic trace vector for N then the map $J_1 : x\xi_0 \to x^*\xi_0$ extends to a conjugation of H such that $J_1 N J_1 = N'$, [4, Ch. I, § 6, Théorème 2].

Lemma 1.4. Let ξ_0 be a separating and cyclic trace vector for N, and let J and J_1 be the conjugations defined above. Let $X = \{\xi \in H : J\xi = \xi\}$. Then we have:

(1) $X = \overline{\mathcal{R}\xi_0}$.

(2) $JJ_1 = J_1J$.

(3) if $x,y \in \mathcal{R}$ and $\xi \in X$ then both $xJyJ\xi$ and $xJ_1yJ_1\xi$ belong to X.

(4) if $x_1,\ldots,x_n$, $y_1,\ldots,y_n \in \mathcal{R}$, $x = \sum_{i=1}^{n} x_iJ_1y_iJ_1$, and $\lambda \in Sp(x) \cap \mathbb{R}$, then given $\epsilon > 0$ there exists $\xi \in \mathcal{R}\xi_0$, $\|\xi\| = 1$, such that $\|x\xi - \lambda\xi\| < \epsilon$.

Proof. Clearly $\overline{\mathcal{R}\xi_0} \subset X$. Conversely, let $\xi \in X$ and let (x_n) be a sequence in N such that $x_n\xi_0 \to \xi$. Then $Jx_n\xi_0 \to J\xi = \xi$, so that $\frac{1}{2}(x_n + Jx_nJ)\xi_0 \to \xi$. Since $x_n + Jx_nJ \in \mathcal{R}$, (1) follows.

If $x \in \mathcal{R}$ then $J_1x\xi_0 = x^*\xi_0 \in \mathcal{R}\xi_0$. Thus $JJ_1x\xi_0 = J_1x\xi_0 = J_1Jx\xi_0$, and since the linear span of $\mathcal{R}\xi_0$ is dense in H, (2) follows.

Let $x,y \in \mathcal{R}$, $\xi \in X$. By (1) $J(xJyJ)\xi = JxJy\xi = Jxy\xi = xy\xi = xJyJ\xi$, so that $xJyJ\xi \in X$. To show that $xJ_1yJ_1\xi \in X$ we may by (1) assume $\xi = w\xi_0$ with $w \in \mathcal{R}$. Since we have

$$\begin{aligned} JxJ_1yJ_1w\xi_0 &= JxJ_1yw^*\xi_0 = Jxwy^*\xi_0 \\ &= xwy^*\xi_0 = xJ_1yw^*\xi_0 \\ &= xJ_1yJ_1w\xi_0, \end{aligned}$$

(3) follows.

Finally let $x = \Sigma x_i J_1 y_i J_1$ be defined as in (4). By the second statement in (3) $JxJ = x$. If λ and $\varepsilon > 0$ are as in (4) there is $\eta \in N\xi_0$, $\|\eta\| = 1$, such that

$$\|x\eta - \lambda\eta\| < \varepsilon/\sqrt{2}.$$

Since $Jx = xJ$ and λ is real we also have

$$\|xJ\eta - \lambda J\eta\| < \varepsilon/\sqrt{2}$$

Let $\theta \in [0,2\pi)$ and $\eta(\theta) = e^{i\theta}\eta$. Then the same inequalities hold for $\eta(\theta)$ instead of η. Let

$$\xi(\theta) = \tfrac{1}{2}(\eta(\theta) + J\eta(\theta)) = \tfrac{1}{2}(e^{i\theta}\eta + e^{-i\theta}J\eta).$$

Then

$$\|\xi(\theta)\|^2 = \tfrac{1}{2}(\|\eta\|^2 + \mathrm{Re}\, e^{i2\theta}(\eta, J\eta)),$$

hence if we choose θ so that $\operatorname{Re} e^{i2\theta}(\eta, J\eta) \geq 0$, we have $\|\xi(\theta)\| \geq \frac{1}{\sqrt{2}}\|\eta\| = 1/\sqrt{2}$. Let ξ' be this $\xi(\theta)$. Then $1 \geq \|\xi'\| \geq 1/\sqrt{2}$, and $\xi' \in X$ since $\psi + J\psi \in X$ for all $\psi \in H$. Since also $\xi' \in N\xi_o$ we have $\xi' \in \mathcal{R}\xi_o$. Finally we have

$$\begin{aligned} \|x\xi' - \lambda\xi'\| &= \|(x-\lambda 1)\tfrac{1}{2}(\eta(\theta) + J\eta(\theta))\| \\ &\leq \tfrac{1}{2}\|(x-\lambda 1)\eta(\theta)\| + \tfrac{1}{2}\|(x-\lambda 1)J\eta(\theta)\| \\ &< \tfrac{1}{2}\,\varepsilon/\sqrt{2} + \tfrac{1}{2}\,\varepsilon/\sqrt{2} = \varepsilon/\sqrt{2}. \end{aligned}$$

Since $\|\xi'\| \geq 1/\sqrt{2}$, (4) follows with $\xi = \|\xi'\|^{-1}\xi'$. Q.E.D.

Remark 1.5. Let $x \in \mathcal{R}$ and $x = u(x)|x|$ be its polar decomposition with $u(x) \in \mathcal{R}$ as in Lemma 1.1. Let τ be a faithful normal semi-finite trace on N. Following Connes [3] we let E_a denote the characteristic function of the open interval $(a, +\infty)$ when $a > 0$, and let $u_a(x) = u(x)E_a(|x|)$. Since by spectral theory $E_a(|x|) \in \mathcal{R}$, so does $u_a(x)$. Thus [3, Theorem 1.2.2] and its corollary [3, Corollary 1.2.3] hold for operators in $\mathcal{R}$ as well. We shall therefore refer to these two results freely.

We let $\operatorname{Aut} N$ denote the automorphism group of N equipped with the topology of strong pointwise convergence. We let $\operatorname{Int} N$ denote the group of inner *-automorphisms of N and $\overline{\operatorname{Int}}\, N$ its closure in $\operatorname{Aut} N$. It is quite easy to show that if $\alpha \in \operatorname{Int} N$ and $\alpha(\mathcal{R}) = \mathcal{R}$, then there is a unitary $u \in \mathcal{R}$ such that $\alpha = \operatorname{Ad} u$. We shall not need this result, but we will need its analogue for $\alpha \in \overline{\operatorname{Int}}\, N$. This follows from the following modification of [3, Theorem 3.1].

Theorem 1.6. Assume N is a factor of type II_1 with separable predual and separating and cyclic trace vector ξ_o. Let $\theta \in \operatorname{Aut} N$

satisfy $\theta(\mathcal{R}) = \mathcal{R}$. Then the following conditions are equivalent.

(1) $\theta \in \overline{\mathrm{Int}}\, N$

(2) There is an automorphism of the C^*-algebra generated by N and N' which is θ on N and the identity on N'.

(3) For any unitaries $u_1,\dots,u_n \in \mathcal{R}$ and $\epsilon > 0$ there is $\xi \in \mathcal{R}\xi_o$, $\|\xi\| = 1$, such that $\|\theta(u_k)J_1 u_k J_1 \xi - \xi\| < \epsilon$ for $k = 1,\dots,n$.

(4) There is a bounded sequence (x_n) in $\mathcal{R}$ not converging strongly to 0 such that $x_n a - \theta(a)x_n \to 0$ strongly for all $a \in N$.

(5) There is a sequence (v_n) of unitaries in $\mathcal{R}$ such that $\theta = \lim\limits_n \mathrm{Ad}\, v_n$ in $\mathrm{Aut}\, N$.

The proof is a direct modification of that of Connes. The only part that requires an additional argument is that of (2) => (3). But in the proof of (b) => (c) in [3] the operator S constructed is of the form $\Sigma x_i J_1 y_i J_1$, $x_i y_i \in \mathcal{R}$, so Lemma 1.4 (4) provides the desired vector $\xi \in \mathcal{R}\xi_o$.

2. Hyperfinite real factors.

Definition. A real von Neumann algebra $\mathcal{R}$ is called a real factor if its center is the real scalar operators. $\mathcal{R}$ is said to be hyperfinite if there exists an increasing sequence R_n of finite dimensional real von Neumann subalgebras R_n of $\mathcal{R}$ with $1 \in R_n$ such that $\bigcup_{n\geq 1} R_n$ is weakly dense in $\mathcal{R}$.

Our main result in the present section is that if $\mathcal{R}$ is a real factor and $N = \mathcal{R} + i\mathcal{R}$ is the hyperfinite II_1-factor, then $\mathcal{R}$ is hyperfinite. We first show there is a unique real hyperfinite factor by proving the analogue of the classical result of Murray and von Neumann, which implies the uniqueness of the hyperfinite II_1-factor [4, Ch III, § 7, Théorème 3]. Recall that if τ is a trace on a finite factor then $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$.

<u>Theorem 2.1.</u> Let N be a factor of type II_1 and $\mathcal{R}$ a real factor such that $N = \mathcal{R} + i\mathcal{R}$. Then the following three conditions are equivalent.

(1) $\mathcal{R}$ is hyperfinite.

(2) $\mathcal{R}$ is the weak closure of the union of an increasing sequence $\{R_n\}$ of real factors with $1 \in R_n$, such that R_n is isomorphic to the real $2^n \times 2^n$ matrices.

(3) $\mathcal{R}$ is countably generated, and given $x_1, \dots, x_n \in \mathcal{R}$ and $\varepsilon > 0$ there exist a finite dimensional real von Neumann subalgebra B of $\mathcal{R}$ and $y_1, \dots, y_n \in B$ such that $\|y_k - x_k\|_2 < \varepsilon$, $k = 1, \dots, n$.

We indicate the proof by referring to the different lemmas in [4, Ch. III, § 7], which are used to prove [4, Ch. III, § 7, Théorème 3].

Lemma 4 is just spectral theory, hence holds for $\mathcal{R}$. Lemma 5 uses polar decomposition, hence follows from Lemma 1.1. Lemma 6 is identical, while Lemma 7 follows because of Lemma 1.3. The crucial part is the modification of Lemma 8. It is a consequence of the following observation, which is a direct consequence of [9, Theorem 3.7] and the classification of irreducible JW-algebras in [7].

Lemma 2.2. Let $\mathcal{R}$ be an irreducible real von Neumann algebra acting on the n-dimensional Hilbert space $\mathbb{C}^n$, $n < \infty$. Then there exists a conjugate linear isometry J on $\mathbb{C}^n$ such that $JxJ = x$ for all $x \in \mathcal{R}$, and either $J^2 = 1$ or $J^2 = -1$. Furthermore if $J^2 = 1$ $\mathcal{R}$ is isomorphic to the real $n \times n$ matrices, and if $J^2 = -1$ then n is even and $\mathcal{R}$ is isomorphic to the $n/2 \times n/2$ matrices over the quaternions.

As an immediate corollary we have

Corollary 2.3. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be irreducible real von Neumann algebras acting on the finite dimensional Hilbert spaces $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$ respectively. Let J_k be the conjugate linear isometry on $\mathbb{C}^{n_k}$ associated with $\mathcal{R}_k$ as in Lemma 2.2, $k = 1,2$. Let $\mathcal{R}_1 \otimes \mathcal{R}_2$ be the tensor product of $\mathcal{R}_1$ and $\mathcal{R}_2$ acting on $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$. Then the conjugate linear isometry associated with $\mathcal{R}_1 \otimes \mathcal{R}_2$ is $J_1 \otimes J_2$, hence we have:

(1) If $J_k^2 = +1$ or $J_k^2 = -1$ for both $k = 1,2$, then $\mathcal{R}_1 \otimes \mathcal{R}_2$ is isomorphic to the real $n_1 n_2 \times n_1 n_2$ matrices.

(2) If $J_1^2 = +1$ and $J_2^2 = -1$, or $J_1^2 = -1$ and $J_2^2 = +1$, then $\mathcal{R}_1 \otimes \mathcal{R}_2$ is isomorphic to the $n_1 n_2/2 \times n_1 n_2/2$ matrices over the quaternions.

The modified version of Lemma 8 states that if $\mathcal{R}$ satisfies condition (3) in Theorem 2.1, and $x_1, \ldots, x_m$ belong to $\mathcal{R}$ and $\varepsilon > 0$, then there exist a real subfactor $\mathcal{B}$ of $\mathcal{R}$ isomorphic to the real $2^k \times 2^k$ matrices for some k and $y_1, \ldots, y_m \in \mathcal{R}$ such that $\|y_j - x_j\|_2 < \varepsilon$ for $j = 1, \ldots, m$. By a modification of the proof of Lemma 8 there is real subfactor B of $\mathcal{R}$ isomorphic to an irreducible real factor acting on the 2^n-dimensional complex Hilbert space for some n having the required approximative properties. If B is isomorphic to the real $2^n \times 2^n$ matrices we are through. If not, we can use Lemma 1.2 twice on the relative commutant of B in $\mathcal{R}$ to find a real subfactor R_4 of $\mathcal{R} \cap B'$ isomorphic to the real 4×4 matrices. By Corollary 2.3 $R_4 \cong Q \otimes Q$, where Q denotes the quarternions represented as complex 2×2 matrices. If we let $\mathcal{B} = B \otimes Q$ identified with a real subfactor of $\mathcal{R}$, then by Corollary 2.3 $\mathcal{B}$ is isomorphic to the real $2^k \times 2^k$ matrices for some k. Thus the modified Lemma 8 follows.

As the proofs of Lemmas 9, 10, 11 are trivially modified the proof of Theorem 2.1 is easily completed.

We next modify the results of McDuff [5] for our purposes. Let ω be a free ultrafilter on the positive integers $\mathbb{N}$, and assume N is a factor of type II_1 with trace τ implemented by a separating and cyclic trace vector ξ_o, and $N = \mathcal{R} + i\mathcal{R}$. Let $\bigoplus_\infty N$ be the direct sum of a countable number of copies of N and similarly define $\bigoplus_\infty \mathcal{R}$. Let

$$I_\omega = \{(t_k) \in \bigoplus_\infty N : \lim_\omega \|t_k\|_2 = 0\}$$

$$J_\omega = \{(r_k) \in \bigoplus_\infty \mathcal{R} : \lim_\omega \|r_k\|_2 = 0\}$$

Then I_ω and J_ω are maximal ideals in $\bigoplus_\infty N$ and $\bigoplus_\infty \mathcal{R}$ respectively, see [5], and $\bigoplus_\infty N / I_\omega = N^\omega$ is a factor of type II_1. Similarly let $\mathcal{R}^\omega = \bigoplus_\infty \mathcal{R} / J_\omega$. Note that $I_\omega = J_\omega + iJ_\omega$. Indeed, let $t_k = r_k + is_k \in \mathcal{R} + i\mathcal{R} = N$, $k \in \mathbb{N}$, and $(t_k) \in I_\omega$. Let J be the conjugation such that $\mathcal{R} = \{x \in N : x = JxJ\}$. Then clearly $(Jt_kJ) \in I_\omega$, hence so is $(\frac{1}{2}(t_k + Jt_kJ)) = (r_k)$. But then $(s_k) = (-i(t_k - r_k)) \in I_\omega$, i.e. (r_k) and (s_k) belong to J_ω, and we have shown $I_\omega \subset J_\omega + iJ_\omega$. Since the converse inclusion is obvious the assertion follows. It follows that the sequence (t_k) is ω-central if and only if (r_k) and (s_k) are ω-central sequences.

Let $\tilde{N} = \{(t_k) \in \bigoplus_\infty N : t_k = t,\ k \in \mathbb{N}\}$. Let ρ denote the canonical homomorphism of $\bigoplus_\infty N$ onto N^ω. Then by the above the canonical map

$$\mathcal{R}^\omega = \bigoplus_\infty \mathcal{R}/J_\omega \to \bigoplus_\infty N/I_\omega = N^\omega$$

is injective since $J_\omega = I_\omega \cap \bigoplus_\infty \mathcal{R}$, and onto $\rho(\bigoplus_\infty \mathcal{R})$. Let now

$$C_N^\omega = N^\omega \cap \rho(\tilde{N})'$$
$$C_{\mathcal{R}}^\omega = \mathcal{R}^\omega \cap \rho(\tilde{N})',$$

where $C_{\mathcal{R}}^\omega$ is considered as a subset of C_N^ω. From the above C_N^ω is a von Neumann algebra generated by the real von Neumann algebra $C_{\mathcal{R}}^\omega$. Modifying the proof of Corollary to Lemma 7 in [5] we have from Lemma 1.2 the following.

<u>Lemma 2.4.</u> If C_N^ω is not commutative then there exists a real subfactor of $C_{\mathcal{R}}^\omega$ which is isomorphic to the real 2×2 matrices.

Let C_N denote the central sequences in N and H_N the hypercentral sequences in N. Then we find as in [5, Theorem 3] the following result.

<u>Theorem 2.5.</u> Suppose N is a factor of type II_1 such that $C_N \neq H_N$. Then $\mathcal{R}$ is isomorphic to $R \otimes \mathcal{R}$, where R is the hyperfinite real factor.

<u>Corollary 2.6.</u> Suppose N is the hyperfinite II_1 factor. Then $\mathcal{R}$ is isomorphic to $R \otimes \mathcal{R}$.

We are now in position to prove the analogue of Connes' fundamental theorem on the hyperfinite II_1-factor [3, Theorem 5.1]. We assume N is the hyperfinite II_1-factor and want to show $\mathcal{R}$ is hyperfinite. I follows from our arguments that all the seven conditions in [3, Theorem 5.1] hold for $\mathcal{R}$ with proper modifications. We first show the proper modification of Condition 6. Recall that $\langle\ ,\ \rangle_{HS}$ (resp. $\|\ \|_{HS}$) denotes the Hilbert-Schmidt inner product (resp. norm) in $B(H)$.

Lemma 2.7. Assume N is the hyperfinite factor acting on the Hilbert space H and having a separating and cyclic trace vector ξ_o. Let τ denote the trace, and let J denote the conjugation such that $x = JxJ$ for $x \in \mathcal{R}$. Let $\mathcal{R}(H) = \{x \in B(H) : x = JxJ\}$. Then if $x_1,\ldots,x_n \in \mathcal{R}$ and $\varepsilon > 0$ there exists a nonzero finite dimensional projection $e \in \mathcal{R}(H)$ such that for all $k = 1,\ldots,n$,

$$\|[x_k,e]\|_{HS} \leq \varepsilon\|e\|_{HS}\ ,\ |\tau(x_k) - \langle x_k e,e\rangle_{HS}/\langle e,e\rangle_{HS}| \leq \varepsilon.$$

Proof. By [3, Theorem 5.1] there exists a state φ on $B(H)$ containing N in its centralizer $B(H)_\varphi$. Since $N = JNJ$, $N \subset B(H)_{J\varphi J}$, where $J\varphi J$ is the state $J\varphi J(x) = \varphi(Jx^*J)$. Let $\omega = \frac{1}{2}(\varphi + J\varphi J)$. Then $\omega = J\omega J$, and $N \subset B(H)_\omega$. The rest of the proof is now a modification of that of (7) => (6) in [3]. Let $F = (u_j)_{j=1,\ldots,p}$ be a finite set of unitaries in $\mathcal{R}$, and let W be the set of all $(\psi - \psi \circ Adu_1,\ldots,\psi - \psi \circ Adu_p)$ for ψ a normal state on $B(H)$ such that $\psi = J\psi J$. As in [3] the existence of ω shows that $(0,\ldots,0) \in W$, so for a suitably small $\eta > 0$ there is a normal state ψ on $B(H)$ such that $\psi = J\psi J$ and $\|\psi - \psi \circ Adu_k\| < \eta$. Let ρ be the unique Hilbert-Schmidt operator such that $\psi(x) = \langle x\rho,\rho\rangle_{HS}$. Since $\psi = J\psi J$, $\rho = J\rho J \in \mathcal{R}(H)$. Since $u_k \in \mathcal{R}$,

$\rho_k = u_k \rho u_k^* \in \mathcal{R}(H)$, so by Remark 1.5 $E_a(\rho)$ and $E_a(\rho_k) \in \mathcal{R}(H)$. Thus $e = E_a(\rho)$ as in Connes' proof is the required projection.

Q.E.D.

We next state the analogue of [3, Lemma 5.25]. Recall that R denotes the hyperfinite real factor.

Lemma 2.8. Assume N is the hyperfinite II_1 factor in standard representation. Then there exists for each free ultrafilter ω on $\mathbb{N}$ a normal isomorphism θ of $N \otimes N$ in the ultraproduct $(N \otimes N)^\omega$ carrying $\mathcal{R} \otimes \mathcal{R}$ into $(\mathcal{R} \otimes R)^\omega$ such that

(1) For each $x \in \mathcal{R}$, $\theta(x \otimes 1)$ is represented by the sequence $(x \otimes 1)_{\nu \in \mathbb{N}}$.

(2) For each $y \in \mathcal{R}$, $\theta(1 \otimes y)$ is represented by a sequence of the form $(1 \otimes z_\nu)_{\nu \in \mathbb{N}}$, $z_\nu \in R$.

The proof consists of a modification of the proofs of lemmas 5.17, 5.22, and 5.25 in [3]. By Lemma 2.7 $\mathcal{R}$ satisfies the real analogue of Condition 6 in [3, Theorem 5.1], hence the real analogue of [3, Lemma 5.17] is immediate, where the finite dimensional factor Q appearing in [3, Lemma 5.17] is replaced by a real factor isomorphic to the real $r \times r$ matrices for some r, and $Q \subset \mathcal{R}(H)$, cf. Lemma 2.7. [3, Lemma 5.22] is changed to the conclusion that there exists a normal homomorphism of $\mathcal{R}$ into R^ω, thus the proof of [3, Lemma 5.25] gives the conclusions of Lemma 2.8.

Theorem 2.9. Let N be the hyperfinite factor of type II_1. Let $\mathcal{R}$ be a real factor such that $N = \mathcal{R} + i\mathcal{R}$. Then $\mathcal{R}$ is hyperfinite.

<u>Proof</u>. We modify the proofs of 3) => 2) => 1) in the proof of [3, Theorem 5.1] and first prove the analogue of 3) => 2). Let $\theta : N \otimes N \to (N \otimes N)^\omega$ be as in Lemma 2.8. Let $x_1, \dots, x_n \in \mathcal{R}$ and $\epsilon > 0$. Since N is hyperfinite the symmetry $\sigma : N \otimes N \to N \otimes N$ carrying $y \otimes x$ onto $y \otimes x$ belongs to $\overline{\mathrm{Int}}\, N \otimes N$. Since clearly $\sigma : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$, and $\mathcal{R} \otimes \mathcal{R}$ is a real von Neumann algebra generating $N \otimes N$, there exists by Theorem 1.6 a unitary $v \in \mathcal{R} \otimes \mathcal{R}$ such that

$$\|x_k \otimes 1 - v(1 \otimes x_k)v^*\|_2 < \epsilon/2, \quad k = 1, \dots, n.$$

As θ preserves the L^2-norm and is a $*$-homomorphism

$$\|\theta(x_k \otimes 1) - \theta(v)\theta(1 \otimes x_k)\theta(v)^*\|_2 < \epsilon/2, \quad k = 1, \dots, n.$$

Let $(X_\nu)_{\nu \in \mathbb{N}}$ be a representing sequence of unitary operators in $\mathcal{R} \otimes \dot{R}$ for $X = \theta(v) \in (\mathcal{R} \otimes R)^\omega$. Let for each k, $(z_k^\nu)_{\nu \in \mathbb{N}}$ be a sequence of elements in R such that $(1 \otimes z_k^\nu)_{\nu \in \mathbb{N}}$ represents $\theta(1 \otimes x_k)$, see Lemma 2.8. Then we have by the above inequality

$$\lim_{\nu \to \omega} \|x_k \otimes 1 - X_\nu(1 \otimes z_k^\nu)X_\nu^*\|_2 \leq \epsilon/2, \quad k = 1, \dots, n,$$

so for a suitable $\nu \in \mathbb{N}$ we have a unitary $X \in \mathcal{R} \otimes R$ and $z_1, \dots, z_n \in R$ such that

$$\|x_k \otimes 1 - X(1 \otimes z_k)X^*\|_2 < \epsilon, \quad k = 1, \dots, n.$$

Since by Corollary 2.6 $\mathcal{R}$ is isomorphic to $\mathcal{R} \otimes R$ a straightforward modification of the proof of 2) => 1) in [3, Theorem 5.1] now shows that condition (3) in Theorem 2.1 is satisfied for $\mathcal{R}$, hence $\mathcal{R}$ is hyperfinite by that theorem. Q.E.D.

Corollary 2.10. There is up to conjugacy a unique involutory $*$-anti-automorphism of the hyperfinite factor of type II_1.

Proof. Let α be an involutory $*$-anti-automorphism of the hyperfinite factor N of type II_1. Let $\mathcal{R} = \{x \in N : \alpha(x^*) = x\}$. Then $\mathcal{R}$ is a real factor such that $\mathcal{R} + i\mathcal{R} = N$, hence by Theorem 2.9 $\mathcal{R}$ is hyperfinite. By Theorem 2.1 there is an increasing sequence of real subfactors (R_n) of $\mathcal{R}$ such that $1 \in R_n$, R_n is isomorphic to the real $2^n \times 2^n$ matrices, and $\bigcup_{n\geq 1} R_n$ is weakly dense in $\mathcal{R}$. If $M_n = R_n + iR_n$ then M_n is a factor of type I_{2^n} such that if t_n denotes the transpose map of M_n with respect to a basis such that $R_n = \{x \in M_n : t_n(x^*) = x\}$, then $\alpha(x) = t_n(x)$ for all $x \in M_n$. Thus α is the inductive limit of the transpose maps t_n. Since any two involutory $*$-anti-automorphisms of N are thus obtained, and every $*$-automorphism of the union $\bigcup_{n\geq 1} M_n$ extends to one of N, [6], they are conjugate by a $*$-automorphism of N. Q.E.D.

References

1) A. Connes, Periodic automorphisms of the hyperfinite factor of type II_1, Acta Sci. Math. 39 (1977), 39-66.

2) ________, Outer conjugacy classes of automorphisms of factors, Ann. Sci. Ec. Norm. Sup. 8 (1975), 383-420.

3) ________, Classification of injective factors, Ann. of Math. 104 (1976), 73-115.

4) J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien. 2nd. Edition, Paris, Gauthier-Villars, 1969.

5) D.McDuff, Central sequences and the hyperfinite factor. Proc. London Math. Soc. XXI (1970), 443-461.

6) C. Pearcy and J. Ringrose, Trace preserving homorphisms in finite operator algebras, Amer. J. Math. 90 (1968), 444-455.

7) E. Størmer, Jordan algebras of type I, Acta math. 115 (1966), 165-184.

8) ________, Irreducible Jordan algebras of self-adjoint operators, Trans. Amer. Math. Soc. 130 (1968), 153-166.

9) ________, On anti-automorphisms of von Neumann algebras, Pacific J. Math. 21 (1967), 349-370.

10) D. Topping, Jordan algebras of self-adjoint operators, Mem. Amer. Math. Soc. 53 (1965).